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Lower bounds for the first eigenvalue of the Dirac operator on compact Riemannian manifolds

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ABSTRACT

Some new, improved, curvature depending lower bounds for the first eigenvalue of the Dirac operator on compact Riemannian manifolds are proved. If certain curvature conditions are satisfied, then these lower bounds are also useful in cases where the scalar curvature has zeros or attains negative values. This implies stronger vanishing theorems for harmonic spinors.

1. Introduction

On any compact n-dimensional Riemannian spin manifold M with scalar curvature $S > 0$, every eigenvalue λ of the Dirac operator D satisfies the inequality

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \cdot S_0, \tag{1}$$

where S_0 denotes the infimum of S on M. This basic result was proved by Th. Friedrich in 1980 [4]. A first generalization of this estimate was given by O. Hijazi [8]. Moreover, in some more special geometric situations, the dimension-depending factor on the right-hand side of (1) can be improved (see [1–3,9,11]). Obviously, this kind of estimates is not useful if S is not positive. Hence, it was of interest to obtain lower bounds for λ^2 that depend on additional curvature terms. Some results in this direction we have proved in [5,6,10]. In [5] and [6], the conditions that the curvature tensor or the Weyl tensor, respectively, is harmonic, play an essential role. [10] contains also results that do not make use of these assumptions. However, one of the problems of our estimates in the general geometric situation is the following. If we specialize these estimates to the case of harmonic curvature tensor or harmonic Weyl tensor, respectively, then we obtain weaker estimates than those we proved before. In this paper we solve this problem by combining the Weitzenböck formulas for the modified twistor operators in a more general way. Using this method we obtain lower bounds that depend on more than one real parameter. Since the optimal parameters can be calculated in any case, we obtain explicit estimates. The new estimates improve the known results if $S_0 \leqslant 0$ but also in some cases with $S_0 > 0$.

Our article is organized as follows. In Section 2, we introduce the basic notions and main identities of Riemannian spin geometry used in this article. In Section 3, we set up the basic Weitzenböck formulas. In Section 4, we prove estimates

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using the assumption that the Weyl tensor or the curvature tensor, respectively, is harmonic. In Section 5, we generalize the results of Section 4 and in Section 6 we show that a further improvement of all estimates is possible. Finally, in Section 7, one finds some remarks on the limiting case of the given estimates.

2. Preliminaries

Let us consider an n-dimensional Riemannian spin manifold M with Riemannian metric g and spinor bundle Σ. We denote by ∇ the covariant derivative induced by g on vector fields as well as on spinor fields (Levi-Civita connection). For any vector fields X and Y, we use the notation

$$\nabla^2_{X,Y} := \nabla_X \cdot \nabla_Y - \nabla_{\nabla_X Y}$$

for the corresponding tensorial derivative of second order. Then, for any vector fields X, Y, Z and any spinor field ψ, the Riemannian curvature tensor R and the corresponding spin curvature tensor C are given by

$$R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z, \qquad C(X,Y)\psi = \nabla^2_{X,Y}\psi - \nabla^2_{Y,X}\psi.$$

If $(X_1, \dots, X_n)$ is any local frame of vector fields, we denote by $(X^1, \dots, X^n)$ the associated coframe defined by $X^k := g^{kl}X_l$, where (g^{kl}) is the inverse of the matrix (g_{kl}) with $g_{kl} := g(X_k, X_l)$. Then, the Ricci tensor Ric, the scalar curvature S and the Dirac operator D are locally given by $\mathrm{Ric}(X) = R(X, X_k)X^k$, $S = \mathrm{tr}(\mathrm{Ric}) = g(\mathrm{Ric}(X_k), X^k)$ and $D\psi = X^k \cdot \nabla_{X_k}\psi$, respectively. (The dot denotes the Clifford multiplication.) The following basic identities of Riemannian spin geometry are well known:

$$C(X,Y) = \frac{1}{4} X_k \cdot R(X,Y)X^k, \tag{2}$$

$$X_k \cdot C\big(X^k, X\big) = \frac{1}{2}\mathrm{Ric}(X) = C\big(X, X^k\big) \cdot X_k, \tag{3}$$

$$X_k \cdot \mathrm{Ric}\big(X^k\big) = -S = \mathrm{Ric}\big(X^k\big) \cdot X_k. \tag{4}$$

The Weyl tensor W and the curvature tensor R are related by

$$W(X,Y)Z = R(X,Y)Z + g\big(K(Y), Z\big)X - g\big(K(X), Z\big)Y + g(Y,Z)K(X) - g(X,Z)K(Y)$$

with $K := \frac{1}{n-2}\big(\frac{S}{2(n-1)} - \mathrm{Ric}\big)$. Using the notation $B(X,Y) := \frac{1}{4}X_k \cdot W(X,Y)X^k$, we have the relation

$$B(X,Y) = C(X,Y) - \frac{1}{2}\big(X \cdot K(Y) - Y \cdot K(X)\big). \tag{5}$$

By (3) this implies the identities

$$X_k \cdot B\big(X^k, X\big) = 0 = B\big(X, X^k\big) \cdot X_k. \tag{6}$$

The curvature endomorphisms $C(X,Y)$ and $B(X,Y)$ are anti-selfadjoint with respect to the Hermitian scalar product $\langle \cdot, \cdot \rangle$ on the spinor bundle Σ, i.e., we have

$$C(X,Y)^* = -C(X,Y), \qquad B(X,Y)^* = -B(X,Y). \tag{7}$$

Thus, the endomorphisms $C^2(X,Y) := C(Y, X_k) \circ C(X^k, X)$, $B^2(X,Y) := B(Y, X_k) \circ B(X^k, X)$ have the property

$$C^2(X,Y)^* = C^2(Y,X), \qquad B^2(X,Y)^* = B^2(Y,X). \tag{8}$$

Hence, the endomorphisms $G := C^2(X_k, X^k)$ and $H := B^2(X_k, X^k)$ of Σ are selfadjoint and nonnegative

$$G^* = G, \qquad H^* = H, \quad G \geqslant 0, \; H \geqslant 0. \tag{9}$$

Using (5) a straightforward calculation shows that G and H are related by

$$G = H + \frac{1}{8}\big(|R|^2 - |W|^2\big) = H + \frac{1}{2(n-2)}\left|\mathrm{Ric} - \frac{S}{n}\right|^2 + \frac{S^2}{4n(n-1)}. \tag{10}$$

Introducing the notations $\delta R(X) := (\nabla_{X_k}R)(X, X^k)$, $\delta W(X) := (\nabla_{X_k}W)(X, X^k)$, $\delta C(X) := (\nabla_{X_k}C)(X, X^k)$, $\delta B(X) := (\nabla_{X_k}B)(X, X^k)$ we have the identities

$$\delta C(X) = \frac{1}{4}X_k \cdot \delta R(X)X^k, \qquad \delta B(X) = \frac{1}{4}X_k \cdot \delta W(X)X^k, \tag{11}$$

$$\delta B(X) = \delta C(X) + \frac{1}{8(n-1)}(X \cdot \nabla S - \nabla S \cdot X), \tag{12}$$

where $\nabla S = g^{-1}(dS)$ is the gradient of the scalar curvature S (here and in the following we also consider the metric g as an isomorphism between the tangent bundle TM and the cotangent bundle T^*M). By the second Bianchi identity it holds that

$$g\big(\delta R(X)Y, Z\big) = g\big((\nabla_Y \operatorname{Ric})Z - (\nabla_Z \operatorname{Ric})Y, X\big). \tag{13}$$

Inserting this into (11) we find

$$\delta C(X) = \frac{1}{4}\big(X^k \cdot (\nabla_{X_k}\operatorname{Ric})X - (\nabla_{X_k}\operatorname{Ric})X \cdot X^k\big). \tag{14}$$

Using (12) and (14) we obtain the equations

$$X_k \cdot \delta C\big(X^k\big) = \frac{1}{4}\nabla S, \qquad X_k \cdot \delta B\big(X^k\big) = 0. \tag{15}$$

For any vector field $X \in \Gamma(TM)$, the endomorphisms $\delta C(X)$ and $\delta B(X)$ of Σ are anti-selfadjoint

$$\delta C(X)^* = -\delta C(X), \qquad \delta B(X)^* = -\delta B(X). \tag{16}$$

This implies that the endomorphisms E, F of Σ, locally defined by $E := -\delta C(X_k) \circ \delta C(X^k)$ and $F := -\delta B(X_k) \circ \delta B(X^k)$, respectively, are selfadjoint and nonnegative

$$E^* = E, \qquad F^* = F, \quad E \geqslant 0,\ F \geqslant 0. \tag{17}$$

By (12) and (15), the equation

$$E = F + \frac{1}{16(n-1)}|\nabla S|^2 \tag{18}$$

is valid. We also use the endomorphism

$$\tilde{E} := E - \frac{1}{16n}|\nabla S|^2 \overset{(18)}{=} F + \frac{1}{16n(n-1)}|\nabla S|^2.$$

Obviously, $\tilde{E}$ is selfadjoint and nonnegative

$$\tilde{E}^* = \tilde{E}, \quad \tilde{E} \geqslant 0. \tag{19}$$

3. Weitzenböck formulas

If M is an n-dimensional Riemannian spin manifold with metric g and spinor bundle Σ, then the associated twistor operator

$$\mathcal{D} : \Gamma(\Sigma) \to \Gamma(TM) \otimes \Sigma)$$

is locally defined by $\mathcal{D}\psi := X^k \otimes \mathcal{D}_{X_k}\psi$, where $(X_1, \dots, X_n)$ is any local frame of vector fields and

$$\mathcal{D}_X\psi := \nabla_X\psi + \frac{1}{n}X \cdot D\psi$$

for every vector field X. The image of $\mathcal{D}$ is contained in the kernel of the Clifford multiplication, i.e., locally we have the equation

$$X^k \cdot \mathcal{D}_{X_k}\psi = 0 \tag{20}$$

for all $\psi \in \Gamma(\Sigma)$. Moreover, for all spinor fields ψ, the well-known Weitzenböck formula

$$|\mathcal{D}\psi|^2 = |\nabla\psi|^2 - \frac{1}{n}|D\psi|^2 \tag{21}$$

is valid, where the function $|\mathcal{D}\psi|^2$ is locally defined by $|\mathcal{D}\psi|^2 := \langle \mathcal{D}_{X_k}\psi, \mathcal{D}_{X^k}\psi\rangle$. A further basic ingredient is the Schrödinger–Lichnerowicz formula

$$\nabla^*\nabla = D^2 - \frac{S}{4}, \tag{22}$$

where the Bochner Laplacian $\nabla^*\nabla : \Gamma(\Sigma) \to \Gamma(\Sigma)$ is locally given by

$$\nabla^*\nabla\psi = -\nabla^2_{X_k, X^k}\psi.$$

For all $t \in \mathbb{R}$, we consider now the differential operators of first order

$$\mathcal{P}^t, \mathcal{Q}^t, \mathcal{R}^t, \mathcal{S}^t : \Gamma(\Sigma) \to \Gamma(TM \otimes \Sigma)$$

defined by

$$\mathcal{P}^t_X \psi := \mathcal{D}_X \psi - t\delta B(X) \cdot \psi,$$
$$\mathcal{Q}^t_X \psi := \mathcal{D}_X \psi - tB\big(X, X^k\big) \cdot \nabla_{X_k} \psi,$$
$$\mathcal{R}^t_X \psi := \nabla_X \psi - t\left(\delta C(X) + \frac{1}{4n} X \cdot \nabla S\right) \cdot \psi,$$
$$\mathcal{S}^t_X \psi := \nabla_X \psi - tC\big(X, X^k\big) \cdot \nabla_{X_k} \psi.$$

By (6), (15) and (20), we see that the images of the operators $\mathcal{P}^t$ and $\mathcal{Q}^t$ are contained in the kernel of the Clifford multiplication, i.e., for all $\psi \in \Gamma(\Sigma)$ and any local frame of vector fields $(X_1, \ldots, X_n)$, we have the equations

$$X^k \cdot \mathcal{P}^t_{X_k} \psi = 0, \qquad X^k \cdot \mathcal{Q}^t_{X_k} \psi = 0. \tag{23}$$

Moreover, we see by (3) and (15) that

$$X^k \cdot \mathcal{S}^t_{X_k} \psi = D\psi - \frac{t}{2} \operatorname{Ric}\big(X^k\big) \cdot \nabla_{X_k} \psi, \tag{24}$$
$$X^k \cdot \mathcal{R}^t_{X_k} \psi = D\psi. \tag{25}$$

Now, we formulate our first proposition, which contains the basic Weitzenböck formulas of this paper. Here, for any $\psi \in \Gamma(\Sigma)$, we use the short hand notations V^B_ψ, V^C_ψ for the vector fields locally given by

$$V^B_\psi := \operatorname{Re}\big(\big\langle B\big(X^k, X^l\big) \cdot \nabla_{X_l} \psi, \psi\big\rangle\big) X_k,$$
$$V^C_\psi := \operatorname{Re}\big(\big\langle C\big(X^k, X^l\big) \cdot \nabla_{X_l} \psi, \psi\big\rangle\big) X_k,$$

respectively, where $\operatorname{Re}(\cdot)$ denotes the real part.

Proposition 3.1. *For all $\psi \in \Gamma(\Sigma)$ and $t \in \mathbb{R}$, the following equations are valid*:

$$\big|\mathcal{P}^t \psi\big|^2 = |\mathcal{D}\psi|^2 + 2t \operatorname{Re}\big(\big\langle \delta B\big(X^k\big) \cdot \nabla_{X_k} \psi, \psi\big\rangle\big) + t^2 \langle F\psi, \psi\rangle, \tag{26}$$
$$\big|\mathcal{Q}^t \psi\big|^2 = |\mathcal{D}\psi|^2 - t\big\langle H\psi, \psi\big\rangle - 2t \operatorname{Re}\big(\big\langle \delta B\big(X^k\big) \cdot \nabla_{X_k} \psi, \psi\big\rangle\big) + t^2\big\langle B^2\big(X^k, X^l\big) \cdot \nabla_{X_k} \psi, \nabla_{X_l} \psi\big\rangle - 2t \operatorname{div}\big(V^B_\psi\big), \tag{27}$$
$$\big|\mathcal{R}^t \psi\big|^2 = |\nabla\psi|^2 + 2t \operatorname{Re}\big(\big\langle \delta C\big(X^k\big) \cdot \nabla_{X_k} \psi, \psi\big\rangle\big) + t^2 \langle \tilde{E}\psi, \psi\rangle + \frac{t}{2n} \operatorname{Re}\big(\big\langle D\psi, \nabla S \cdot \psi\big\rangle\big), \tag{28}$$
$$\big|\mathcal{S}^t \psi\big|^2 = |\nabla\psi|^2 - t\langle G\psi, \psi\rangle - 2t \operatorname{Re}\big(\big\langle \delta C\big(X^k\big) \cdot \nabla_{X_k} \psi, \psi\big\rangle\big) + t^2\big\langle C^2\big(X^k, X^l\big) \cdot \nabla_{X_k} \psi, \nabla_{X_l} \psi\big\rangle - 2t \operatorname{div}\big(V^C_\psi\big). \tag{29}$$

This proposition is proved by straightforward calculations, which we omit here.

4. Eigenvalue estimates in the harmonic cases

In this section, the assumptions $\delta W = 0$ and $\delta R = 0$ play an essential role. It is well known that $\delta W = 0$ is equivalent to the condition that the equation

$$(\nabla_X \operatorname{Ric})Y - (\nabla_Y \operatorname{Ric})X = \frac{1}{2(n-1)}\big(X(S)Y - Y(S)X\big) \tag{30}$$

is valid for all vector fields X, Y and that $\delta R = 0$ is equivalent to the symmetry property

$$(\nabla_X \operatorname{Ric})Y = (\nabla_Y \operatorname{Ric})X \tag{31}$$

of the covariant derivative $\nabla \operatorname{Ric}$. Condition (31) implies that the scalar curvature S is constant. Thus, $\delta R = 0$ is equivalent to $\delta W = 0$, $dS = 0$. Let us consider the endomorphism-valued "3-form" dW defined by

$$dW(X, Y, Z) := (\nabla_X W)(Y, Z) + (\nabla_Y W)(Z, X) + (\nabla_Z W)(X, Y).$$

Then we know that the corresponding 3-form dR vanishes identically due to the Bianchi identity. Moreover, it is known that $\delta W = 0$ implies $dW = 0$. Therefore, the case $\delta W = 0(\delta R = 0)$ is also called the case with harmonic Weyl tensor (harmonic curvature tensor). If the condition $\delta W = 0(\delta R = 0)$ is satisfied, then we also say that the underlying Riemannian manifold

M is W-harmonic (R-harmonic). Obviously, manifolds with parallel Ricci tensor and Einstein manifolds are R-harmonic and, hence, W-harmonic. Further examples are listed in [5], Section 2.

In the following let M be compact. Let us denote the infima of all eigenvalues of the endomorphisms G and H by $\gamma_0 \geqslant 0$ and $\nu_0 \geqslant 0$, respectively. Then we have the inequalities

$$\gamma_0 |\psi|^2 \leqslant \langle G\psi, \psi\rangle, \tag{32}$$

$$\nu_0 |\psi|^2 \leqslant \langle H\psi, \psi\rangle \tag{33}$$

for all $\psi \in \Gamma(\Sigma)$. We see by (10) that γ_0 and ν_0 are related by

$$\gamma_0 \geqslant \nu_0 + \frac{1}{2(n-2)} \left|\mathrm{Ric} - \frac{S}{n}\right|_0^2 + \frac{|S|_0^2}{4n(n-1)}, \tag{34}$$

where the short hand notations

$$\left|\mathrm{Ric} - \frac{S}{n}\right|_0 := \inf_M\left(\left|\mathrm{Ric} - \frac{S}{n}\right|\right), \qquad |S|_0 := \inf_M\bigl(|S|\bigr)$$

are used. Furthermore, we denote the suprema of all eigenvalues of the endomorphisms $\tilde{E}$ and F by $\vartheta \geqslant 0$, $\eta \geqslant 0$, respectively. Then the estimates

$$\langle \tilde{E}\psi, \psi\rangle \leqslant \vartheta |\psi|^2, \tag{35}$$

$$\langle F\psi, \psi\rangle \leqslant \eta |\psi|^2 \tag{36}$$

are valid for all spinor fields ψ. Finally, we introduce the curvature invariants $\mu \geqslant 0$, $\zeta \geqslant 0$. μ is the conformal invariant defined by

$$\mu := \sup\bigl\{\bigl\|B(X,Y)\bigr\| \mid x \in M,\ X, Y \in T_xM,\ |X| = |Y| = 1,\ X \perp Y\bigr\},$$

where $\|\cdot\|$ denotes the operator norm here. ζ is the corresponding supremum if B is replaced by the spin curvature tensor C.

Lemma 4.1. *For all $\psi \in \Gamma(\Sigma)$, we have the estimates*

$$\bigl|\bigl\langle C^2\bigl(X^k, X^l\bigr) \cdot \nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle\bigr| \leqslant (n-1)^2\zeta^2|\nabla\psi|^2, \tag{37}$$

$$\bigl|\bigl\langle B^2\bigl(X^k, X^l\bigr) \cdot \nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle\bigr| \leqslant (n-1)^2\mu^2|\mathcal{D}\psi|^2. \tag{38}$$

For the proof of this lemma we refer to the proof of Lemma 2.2 in [10]. We see by (21) that (38) improves the corresponding estimate (33) in [10]. This improvement was possible by the simple remark that, because of (6), the equation

$$\bigl\langle B^2\bigl(X^k, X^l\bigr) \cdot \nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle = \bigl\langle B^2\bigl(X^k, X^l\bigr) \cdot \mathcal{D}_{X_k}\psi, \mathcal{D}_{X_l}\psi\bigr\rangle \tag{39}$$

is valid.

Lemma 4.2. *If λ is any eigenvalue of the Dirac operator D on a compact n-dimensional W-harmonic Riemannian spin manifold M, then we have the inequality*

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + \frac{4\nu_0 t}{1 + (n-1)^2\mu^2 t^2}\right) \tag{40}$$

for all real parameters $t \geqslant 0$.

Proof. Let λ be any eigenvalue of D and let ψ be any corresponding eigenspinor ($D\psi = \lambda\psi$, $\psi \neq 0$). Inserting ψ into (27) and using the assumption $\delta W = 0$ ($\Rightarrow \delta B = 0$) we obtain

$$\bigl|Q^t\psi\bigr|^2 = |\mathcal{D}\psi|^2 - t\langle H\psi, \psi\rangle - 2t\,\mathrm{div}\bigl(V_\psi^B\bigr) + t^2\bigl\langle B^2\bigl(X^k, X^l\bigr) \cdot \nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle.$$

Now, we integrate this equation using (21), (22), (33), (38). Then we obtain

$$0 \leqslant \left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4} - \nu_0 t + (n-1)^2\mu^2 t^2\left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4}\right)\right) \cdot \int_M |\psi|^2$$

with $S_0 := \inf_M(S)$. This implies

$$\left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4}\right)\left(1+(n-1)^2\mu^2t^2\right) - \nu_0 t \geqslant 0$$

and, hence, the inequality (40). □

Inserting the optimal parameter

$$t_1 = \frac{1}{(n-1)\mu} \tag{41}$$

into (40) we obtain the following result.

Theorem 4.1. *Every eigenvalue λ of the Dirac operator on a compact W-harmonic Riemannian n-manifold satisfies the estimate*

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + \frac{2\nu_0}{(n-1)\mu}\right). \tag{42}$$

We compare this estimate with the estimate

$$\lambda^2 \geqslant \frac{1}{8(n-1)}\left((2n-1)S_0 + \sqrt{S_0^2 + \frac{n}{n-1}\left(\frac{4\nu_0}{\mu}\right)^2}\right), \tag{43}$$

which is given by Corollary 4.1 in [10] that uses the same assumptions as our Theorem 4.1. First we remark that in case $S_0 \leqslant 0$, the lower bound in (42) and also the lower bound in (43) are positive if the condition

$$\nu_0 + \frac{n-1}{2}\mu S_0 > 0 \tag{44}$$

is satisfied. Moreover, a simple calculation shows that the lower bound in (42) is greater than that in (43) if it holds that

$$\nu_0\left(\nu_0 + \frac{n-1}{2}\mu S_0\right) > 0.$$

Thus, in all those cases with $\nu_0 > 0$ where the lower bounds in (42), (43) are positive, the estimate (42) is better than (43). Some information concerning the invariant ν_0 one finds in [6], Section 3.

Lemma 4.3. *For all eigenvalues λ of D on a compact R-harmonic Riemannian n-manifold and all $t \geqslant 0$, the inequality*

$$\lambda^2 \geqslant \frac{S}{4} + \frac{\gamma_0 t}{1+(n-1)^2\zeta^2t^2} \tag{45}$$

is satisfied.

Proof. Let λ be an eigenvalue of D and let ψ be a corresponding eigenspinor. We insert ψ into (29) and then we take the integral of both sides of this equation using the assumption $\delta R = 0$ ($\Rightarrow \delta C = 0$, $dS = 0$), the Schrödinger–Lichnerowicz formula (22) and the estimates (32), (37). This yields the estimate (45). □

The optimal parameter t for the inequality (45) is given by

$$t_2 := \frac{1}{(n-1)\zeta}. \tag{46}$$

Inserting $t = t_2$ into (45) we obtain our next main result.

Theorem 4.2. *Every eigenvalue λ of the Dirac operator on a compact R-harmonic Riemannian n-manifold satisfies the estimate*

$$\lambda^2 \geqslant \frac{S}{4} + \frac{\gamma_0}{2(n-1)\zeta}. \tag{47}$$

Let us compare our Theorem 4.2 with Theorem 5.2 in [10]. In case of $\delta R = 0$, this Theorem 5.2 yields the estimate

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S + t\cdot\frac{\gamma_3(t)}{\alpha_3(t)}\right), \tag{48}$$

for all $t \geqslant 0$, where the functions $\alpha_3(t)$ and $\gamma_3(t)$ are defined by

$$\alpha_3(t) := 1 + \frac{S}{n(n-1)}t + \left(\frac{1}{2n(n-1)}\left|\mathrm{Ric} - \frac{S}{n}\right|_1^2 + 2n(n-1)\zeta^2\right)t^2,$$

$$\gamma_3(t) := 4\gamma_0 - \frac{S^2}{n(n-1)} - S\left(\frac{1}{2n(n-1)}\left|\mathrm{Ric} - \frac{S}{n}\right|_1^2 + 2(n-1)\zeta^2\right)t.$$

Here, the notation $|\mathrm{Ric} - \frac{S}{n}|_1 := \sup_M(|\mathrm{Ric} - \frac{S}{n}|)$ is used.

(At this point we remark that the definition of the functions $\gamma_3(t)$ and $\gamma_4(t)$ in [10], Section 5, is not correct.)

To simplify matters we consider the special case of $S = 0$. In this case, the optimal parameter t in (48) can easily be calculated and we obtain the estimate

$$\lambda^2 \geqslant \frac{n}{2(n-1)} \cdot \frac{\gamma_0}{\sqrt{2n(n-1)\zeta^2 + \frac{|\mathrm{Ric}|_1^2}{2n(n-1)}}}. \tag{49}$$

Obviously, for $S = 0$, the estimate (47) is essentially better than (49). Considering the case that $S \leqslant 0$, we see that the lower bound in (47) is positive if the condition

$$\gamma_0 > \frac{n-1}{2}\zeta|S| \tag{50}$$

is satisfied. However, the lower bound in (48) is positive if the stronger inequality

$$\gamma_0 > \frac{n-1}{2}\zeta|S|\sqrt{2} \tag{51}$$

is valid.

5. Eigenvalue estimates in the general case

In this section we prove eigenvalue estimates without using the assumptions $\delta R = 0$ or $\delta W = 0$, respectively. Such estimates were already given in [10]. However, an essential problem of these estimates is that their application to the special cases $\delta R = 0$ or $\delta W = 0$, respectively, yields weaker results than those was obtained before in the harmonic cases. A further problem of the results in [10], Section 5, is that the given final estimates contain a free parameter since the optimal one leads to a very complicated formula. In this section, we derive results that do not have the mentioned deficiencies.

Lemma 5.1. *If λ is any eigenvalue of the Dirac operator on a compact Riemannian n-manifold, then we have the inequality*

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + 4t\frac{\nu_0 - \eta\frac{t}{r}}{1 + (n-1)^2\mu^2\frac{t^2}{1-r}}\right) \tag{52}$$

for all real parameters $t \geqslant 0$ and $r \in (0, 1)$.

Proof. For any eigenvalue λ of D, any corresponding eigenspinor ψ and any real numbers $s \geqslant 0$, $t \geqslant 0$, $r \in (0, 1)$, it holds that

$$\begin{aligned} r\left|\mathcal{P}^s\psi\right|^2 + (1-r)\left|\mathcal{Q}^t\psi\right|^2 \overset{(26),(27)}{=} {} & |\mathcal{D}\psi|^2 - (1-r)t\langle H\psi, \psi\rangle \\ & + 2\big(rs - (1-r)t\big)\operatorname{Re}\big(\big\langle \delta B\big(X^k\big) \cdot \nabla_{X_k}\psi, \psi\big\rangle\big) + rs^2\langle F\psi, \psi\rangle \\ & + (1-r)t^2\big\langle B^2\big(X^k, X^l\big) \cdot \nabla_{X_k}\psi, \nabla_{X_l}\psi\big\rangle - 2(1-r)t\operatorname{div}\big(V_\psi^B\big). \end{aligned}$$

To remove the term with δB we choose $s = \frac{1-r}{r}t$. If we integrate this equation using (21), (22), (33), (36) and (38), then we obtain

$$0 \leqslant \left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4} - (1-r)t\nu_0 + \frac{(1-r)^2}{r}\eta t^2 + (1-r)t^2(n-1)^2\mu^2\left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4}\right)\right) \cdot \int_M |\psi|^2.$$

This yields

$$\left(\frac{n-1}{n}\lambda^2 - \frac{S_0}{4}\right)\big(1 + (n-1)^2\mu^2(1-r)t^2\big) - \nu_0(1-r)t + \eta\frac{(1-r)^2}{r}t^2 \geqslant 0$$

and, hence, the inequality

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + 4(1-r)t\frac{\nu_0 - \eta\frac{1-r}{r}t}{1+(n-1)^2\mu^2(1-r)t^2}\right),$$

which takes the form (52) if we replace $(1-r)t$ by t. $\square$

Remark 5.1. Our Lemma 5.1 improves the corresponding Theorem 4.2 in [10] in case $S_0 > 0$, since, for $r = \frac{1}{2}$, already the lower bound in (52) is better than the lower bound of the estimate (59) in [10].

Theorem 5.1. *Every eigenvalue λ of the Dirac operator on a compact Riemannian n-manifold satisfies the estimate*

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + \frac{4\nu_0^2}{(\sqrt{\eta + 2(n-1)\mu\nu_0} + \sqrt{\eta})^2}\right). \tag{53}$$

Proof. If we calculate the maximum of the right-hand side of (52) for any fixed $r \in (0,1)$ with respect to $t \geqslant 0$, then we obtain the estimate

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + \frac{2\nu_0^2}{\frac{\eta}{r} + \sqrt{(\frac{\eta}{r})^2 + \frac{(n-1)^2\mu^2\nu_0^2}{1-r}}}\right).$$

Now, the calculation of the minimum of the denominator on the right-hand side with respect to $r \in (0,1)$ yields the estimate (53). $\square$

Remark 5.2. In the W-harmonic case, we have $\eta = 0$. Thus, for $\delta W = 0$, (53) yields exactly the estimate (42).

Remark 5.3. Considering the case that $S_0 \leqslant 0$ we find that the lower bound in (53) is positive if the condition

$$\nu_0 > \frac{n-1}{2}\mu|S_0| + \sqrt{\eta|S_0|} \tag{54}$$

is satisfied. In general, i.e., if

$$|S_0|\left(\eta - \frac{1}{4}(n-1)^2\mu^2|S_0|\right) \neq 0,$$

the condition (54) is weaker than the corresponding condition (61) in [10].

Lemma 5.2. *For any eigenvalue λ of the Dirac operator on a compact Riemannian n-manifold, all $t \geqslant 0$ and $r \in (0,1)$, we have the inequality*

$$\lambda^2 \geqslant \frac{S_0}{4} + t\,\frac{\gamma_0 - \vartheta\frac{t}{r}}{1+(n-1)^2\zeta^2\frac{t^2}{1-r}}. \tag{55}$$

Proof. Let λ be any eigenvalue of D and let ψ be a corresponding eigenspinor. Then, for all $s \geqslant 0, t \geqslant 0$ and $r \in (0,1)$, it holds that

$$\begin{aligned} r\left|\mathcal{R}^s\psi\right|^2 + (1-r)\left|\mathcal{S}^t\psi\right|^2 \overset{(28),(29)}{=} {} & |\nabla\psi|^2 - (1-r)t\langle G\psi,\psi\rangle \\ & + 2\big(rs - (1-r)t\big)\operatorname{Re}\big(\big\langle \delta C\big(X^k\big)\cdot\nabla_{X_k}\psi,\psi\big\rangle\big) + rs^2\langle\tilde{E}\psi,\psi\rangle \\ & + (1-r)t^2\big\langle C^2\big(X^k,X^l\big)\cdot\nabla_{X_k}\psi,\nabla_{X_l}\psi\big\rangle - 2(1-r)t\operatorname{div}\big(V_\psi^C\big). \end{aligned}$$

To remove the term with δC we choose $s = \frac{1-r}{r}t$. Then we integrate this equation and using (22), (32), (35), (37) we obtain

$$0 \leqslant \left(\left(1+(n-1)^2\zeta^2(1-r)t^2\right)\left(\lambda^2 - \frac{S_0}{4}\right) - \gamma_0(1-r)t + \vartheta\frac{(1-r)^2}{r}t^2\right)\cdot\int_M |\psi|^2.$$

This implies the estimate

$$\lambda^2 \geqslant \frac{S_0}{4} + (1-r)t\,\frac{\gamma_0 - \vartheta\frac{1-r}{r}t}{1+(n-1)^2\zeta^2(1-r)t^2}$$

which takes the form (55) if we replace $(1-r)t$ by t. $\square$

Theorem 5.2. *For every eigenvalue λ of the Dirac operator on a compact Riemannian n-manifold, the estimate*

$$\lambda^2 \geqslant \frac{S_0}{4} + \frac{\gamma_0^2}{(\sqrt{\vartheta + 2(n-1)\zeta\gamma_0} + \sqrt{\vartheta})^2} \tag{56}$$

is valid.

Proof. For any fixed $r \in (0,1)$, we calculate the maximum of the lower bound in (55) with respect to $t \geqslant 0$. Then we find the estimate

$$\lambda^2 \geqslant \frac{S_0}{4} + \frac{1}{2}\,\frac{\gamma_0^2}{\frac{\vartheta}{r} + \sqrt{(\frac{\vartheta}{r})^2 + \frac{(n-1)^2\zeta^2\gamma_0^2}{1-r}}},$$

which is valid for all $r \in (0,1)$. Now, calculating the maximum of the right-hand side with respect to r, we obtain the estimate (56). □

Remark 5.4. In the R-harmonic case, we have $\vartheta = 0$, $dS = 0$. Thus, we see that (56) coincides then with (47).

Remark 5.5.

(i) In case $S_0 > 0$, a simple calculation shows that the estimate (56) is better than (1) if the condition

$$\gamma_0 > \frac{1}{2}\zeta S_0 + \sqrt{\frac{\vartheta S_0}{n-1}} \tag{57}$$

is satisfied.

(ii) Considering the case of $S_0 \leqslant 0$, we find that the lower bound in (56) is positive if we have the inequality

$$\gamma_0 > \frac{n-1}{2}\zeta|S_0| + \sqrt{\vartheta|S_0|}. \tag{58}$$

This condition is essentially weaker than the corresponding condition (78) in [10].

By Remark 5.3 and assertion (ii) of Remark 5.5, we immediately obtain the following vanishing theorem for harmonic spinors, which improves the corresponding assertions of Corollaries 4.3 and 5.2 in [10].

Theorem 5.3. *It holds that $\ker(D) = 0$ on a compact Riemannian spin n-manifold with $S_0 \leqslant 0$ if one of the conditions (54) or (58) is satisfied.*

6. A further improvement of the estimates

As before, let M be a compact Riemannian spin n-manifold with metric g and spinor bundle Σ. Then we consider the invariants $b_1 \geqslant 0$, $b_2 \geqslant 0$, $c_1 \geqslant 0$, $c_2 \geqslant 0$ defined by

$$b_1 := \sqrt{\sup\bigl\{\|B^2(X,X)\| \bigm| x \in M,\ X \in T_xM,\ |X| = 1\bigr\}},$$
$$b_2 := \sqrt{\sup\bigl\{\|B^2(X,Y)\| \bigm| x \in M,\ X,Y \in T_xM,\ |X| = |Y| = 1,\ X \perp Y\bigr\}},$$
$$c_1 := \sqrt{\sup\bigl\{\|C^2(X,X)\| \bigm| x \in M,\ X \in T_xM,\ |X| = 1\bigr\}},$$
$$c_2 := \sqrt{\sup\bigl\{\|C^2(X,Y)\| \bigm| x \in M,\ X,Y \in T_xM,\ |X| = |Y| = 1,\ X \perp Y\bigr\}},$$

where $\|\cdot\|$ denotes the operator norm. Moreover, let b and c be the numbers given by

$$b := \sqrt{b_1^2 + (n-1)b_2^2}, \qquad c := \sqrt{c_1^2 + (n-1)c_2^2}.$$

Lemma 6.1. *For every $\psi \in \Gamma(\Sigma)$, the estimates*

$$\bigl|\bigl\langle C^2\bigl(X^k, X^l\bigr)\cdot\nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle\bigr| \leqslant c^2|\nabla\psi|^2, \tag{59}$$
$$\bigl|\bigl\langle B^2\bigl(X^k, X^l\bigr)\cdot\nabla_{X_k}\psi, \nabla_{X_l}\psi\bigr\rangle\bigr| \leqslant b^2|\mathcal{D}\psi|^2, \tag{60}$$

are valid.

Proof. Let us prove the inequality (60). If $(X_1, \dots, X_n)$ is any local orthonormal frame of vector fields, then we have

$$\begin{aligned}
\big|\big\langle B^2\big(X^k, X^l\big)\cdot\nabla_{X_k}\psi, \nabla_{X_l}\psi\big\rangle\big| &\overset{(39)}{=} \big|\big\langle B^2\big(X^k, X^l\big)\cdot\mathcal{D}_{X_k}\psi, \mathcal{D}_{X_l}\psi\big\rangle\big| \\
&\leqslant \sum_{k,l}\big|\big\langle B^2(X_k, X_l)\cdot\mathcal{D}_{X_k}\psi, \mathcal{D}_{X_l}\psi\big\rangle\big| \\
&\leqslant \sum_{k,l}\big|B^2(X_k, X_l)\cdot\mathcal{D}_{X_k}\psi\big||\mathcal{D}_{X_l}\psi| \\
&\leqslant \sum_{k,l}\big\|B^2(X_k, X_l)\big\||\mathcal{D}_{X_k}\psi||\mathcal{D}_{X_l}\psi| \\
&= \sum_{k}\big\|B^2(X_k, X_k)\big\||\mathcal{D}_{X_k}\psi|^2 + \sum_{k\neq l}\big\|B^2(X_k, X_l)\big\||\mathcal{D}_{X_k}\psi||\mathcal{D}_{X_l}\psi| \\
&\leqslant b_1^2\sum_k|\mathcal{D}_{X_k}\psi|^2 + b_2^2\sum_{k\neq l}|\mathcal{D}_{X_k}||\mathcal{D}_{X_l}\psi| \\
&= b_1^2|\mathcal{D}\psi|^2 + b_2^2\sum_{k,l}|\mathcal{D}_{X_k}\psi||\mathcal{D}_{X_l}\psi| - b_2^2\sum_k|\mathcal{D}_{X_k}\psi|^2 \\
&= \big(b_1^2 - b_2^2\big)|\mathcal{D}\psi|^2 + b_2^2\Big(\sum_k|\mathcal{D}_{X_k}\psi|\Big)^2 \\
&\leqslant \big(b_1^2 - b_2^2\big)|\mathcal{D}\psi|^2 + nb_2^2\Big(\sum_k|\mathcal{D}_{X_k}\psi|^2\Big) \\
&= \big(b_1^2 + (n-1)b_2^2\big)|\mathcal{D}\psi|^2 = b^2|\mathcal{D}\psi|^2.
\end{aligned}$$

The proof of (59) is quite analogous. □

Lemma 6.2. *There exist the inequalities*

$$c_1^2 \leqslant (n-1)\zeta^2, \qquad c_2^2 \leqslant (n-2)\zeta^2, \tag{61}$$

$$b_1^2 \leqslant (n-1)\mu^2, \qquad b_2^2 \leqslant (n-2)\mu^2, \tag{62}$$

$$c^2 \leqslant (n-1)^2\zeta^2, \tag{63}$$

$$b^2 \leqslant (n-1)^2\mu^2. \tag{64}$$

Proof. For any point $x \in M$, any orthonormal basis $(X_1, \dots, X_n)$ of T_xM and $k, l \in \{1, \dots, n\}$, it holds that

$$\big\|C^2(X_k, X_l)\big\| = \Big\|\sum_j C(X_l, X_j)\circ C(X_j, X_k)\Big\| \leqslant \sum_j\big\|C(X_l, X_j)\circ(X_j, X_k)\big\| \leqslant \sum_j\big\|C(X_l, X_j)\big\|\big\|C(X_j, X_k)\big\|.$$

Since $\|C(X_j, X_k)\| \leqslant \zeta$ for all $j, k \in \{1, \dots, n\}$, this yields the inequalities

$$\begin{aligned}
&\big\|C^2(X_k, X_k)\big\| \leqslant (n-1)\zeta^2 \quad (k = 1, \dots, n), \\
&\big\|C^2(X_k, X_l)\big\| \leqslant (n-2)\zeta^2 \quad (k, l = 1, \dots, n;\ k \neq l).
\end{aligned}$$

Thus, for all $X, Y, Z \in T_xM$ with $|X| = |Y| = |Z| = 1$ and $Y \perp Z$, it follows that

$$\big\|C^2(X, X)\big\| \leqslant (n-1)\zeta^2, \qquad \big\|C^2(Y, Z)\big\| \leqslant (n-2)\zeta^2.$$

According to the definition of c_1 and c_2 this implies the inequalities (61) and, hence, the inequality (63). Quite analogous considerations yield (62) and (64). □

From Lemmas 6.1 and 6.2 we conclude immediately that the estimates (59), (60) improve the corresponding estimates (37), (38) of Lemma 4.1. Thus, if we use (59), (60) instead of (37), (38), respectively, then we obtain better estimates in general. We omit listing all the results in their improved version. We only remark that one obtains the improved results by simply replacing μ by $\frac{b}{n-1}$ and ζ by $\frac{c}{n-1}$ in all formulas of Sections 4 and 5.

7. Some remarks on the limiting case

A problem connected with the eigenvalue estimates is to study their limiting case. The question is whether the considered estimate is sharp in the sense that there exist manifolds (so-called limiting manifolds) for which the estimate is an equality for the first eigenvalue of the Dirac operator. In the cases with $S_0 \leqslant 0$, we always suppose in the following that the lower bound of the considered estimate is positive.

We start with the improved version of the estimate (56), which is of the form

$$\lambda^2 \geqslant \frac{S_0}{4} + \frac{\gamma_0^2}{(\sqrt{\vartheta + 2c\gamma_0} + \sqrt{\vartheta})^2}. \tag{65}$$

Remark 7.1. For $\vartheta > 0$ ($\delta R \neq 0$), (65) can never be an equality. This follows from the proof of Theorem 5.2 and (25). Thus, every limiting manifold of the estimate (65) and also of the estimate (56) must be R-harmonic.

It remains to study the limiting case of the estimate

$$\lambda^2 \geqslant \frac{S}{4} + \frac{\gamma_0}{2c} \tag{66}$$

for R-harmonic manifolds. Then $\lambda_0 := \sqrt{\frac{S}{4} + \frac{\gamma_0}{2c}}$ is an eigenvalue of D and every associated eigenspinor ψ satisfies the equations

$$D\psi = \lambda_0 \psi, \tag{67}$$

$$\mathcal{S}^t \psi = 0, \tag{68}$$

where the optimal parameter t is given by $t = \frac{1}{c}$ (see (46)) here. We remark that (68) is equivalent to the equation

$$\nabla_X \psi = \frac{1}{c} C\big(X, X^k\big) \cdot \nabla_{X_k} \psi \tag{69}$$

for all vector fields X. We remark further that in the limiting case of (66) every associated eigenspinor ψ also satisfies the eigenvalue equation

$$G\psi = \gamma_0 \psi. \tag{70}$$

Proposition 7.1. *If M is a compact R-harmonic Riemannian spin manifold such that there exists a spinor $0 \neq \psi \in \Gamma(\Sigma)$ that satisfies Eqs. (69) and (70), then M is a limiting manifold of the estimate (66).*

Proof. From (69) we derive

$$\nabla^2_{X,Y} \psi = \frac{1}{c}\big((\nabla_X C)\big(Y, X^k\big) \cdot \nabla_{X_k} \psi + C\big(Y, X^k\big) \cdot \nabla^2_{X, X_k} \psi\big). \tag{71}$$

Using the property $\delta C = 0$ we calculate

$$\begin{aligned} D^2\psi - \frac{S}{4}\psi &\overset{(22)}{=} -\nabla^2_{X_j, X^j}\psi \overset{(71)}{=} -\frac{1}{c} C\big(X^j, X^k\big) \cdot \nabla^2_{X_j, X_k}\psi \\ &= -\frac{1}{2c} C\big(X^j, X^k\big) \circ C(X_j, X_k)\psi = \frac{1}{2c} G\psi \overset{(70)}{=} \frac{\gamma_0}{2c}\psi. \end{aligned}$$

Thus, we obtain

$$D^2\psi = \left(\frac{S}{4} + \frac{\gamma_0}{2c}\right)\psi. \qquad \square$$

Remark 7.2. The simplest example of a compact R-harmonic manifold is a manifold of constant positive curvature ρ, where the curvature tensors R and C are given by

$$R(X, Y)Z = \rho\big(g(Y, Z)X - g(X, Z)Y\big), \tag{72}$$

$$C(X, Y) = \frac{\rho}{4}(Y \cdot X - X \cdot Y). \tag{73}$$

Then the scalar curvature S and ρ are related by

$$S = n(n-1)\rho. \tag{74}$$

Using (73) we find

$$C^2(X, Y) = \frac{\rho^2}{4}\big((2n-3)g(X, Y) + (n-2)X \cdot Y\big). \tag{75}$$

For all X, Y with $|X| = |Y| = 1$ and $X \perp Y$, this yields

$$C^2(X, X) = \frac{n-1}{4}\rho^2, \qquad C^2(X, Y) = \frac{n-2}{4}\rho^2 X \cdot Y \tag{76}$$

and, hence, $c_1 = \frac{\rho}{2}\sqrt{n-1}$, $c_2 = \frac{\rho}{2}\sqrt{n-2}$, $c = \sqrt{c_1^2 + (n-1)c_2^2} = \frac{n-1}{2}\rho$. Thus, using (74) we obtain

$$c = \frac{S}{2n}. \tag{77}$$

Moreover, we see by (10) that we have here

$$\gamma_0 = \frac{S^2}{4n(n-1)}. \tag{78}$$

Using (77), (78) we find that λ_0 in our example is given by

$$\lambda_0 = \sqrt{\frac{n}{4(n-1)}S}. \tag{79}$$

Finally, we see by (73), (74), (77) that Eq. (69) is equivalent to the twistor equation

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0, \tag{80}$$

which, in the limiting case $D\psi = \lambda_0\psi$, implies that every associated eigenspinor is a Killing spinor. This shows that in the simplest case the improved version of Theorem 5.2 (and also Theorem 5.2 itself) reproduces Friedrich's result (1).

Inserting Eq. (69) into itself we find that in the limiting case the eigenspinor ψ also satisfies the equation

$$\nabla_X \psi = \frac{1}{c^2} C^2\big(X^k, X\big) \cdot \nabla_{X_k} \psi. \tag{81}$$

If $(X_1, \dots, X_n)$ is any local orthonormal frame of vector fields, then we obtain by (81) for every $j \in \{1, \dots, n\}$ the equation

$$\nabla_{X_j} \psi = \frac{1}{c^2} \sum_k C^2(X_k, X_j) \cdot \nabla_{X_k} \psi.$$

This yields

$$\begin{aligned}
|\nabla_{X_j} \psi| &\leqslant \frac{1}{c^2} \left(\sum_k \big\| C^2(X_k, X_j) \big\| |\nabla_{X_k} \psi| \right) \\
&= \frac{1}{c^2} \| C^2(X_j, X_j) \| |\nabla_{X_j} \psi| + \frac{1}{c^2} \sum_{k \neq j} \big\| C^2(X_k, X_j) \big\| |\nabla_{X_k} \psi| \\
&\leqslant \frac{c_1^2}{c^2} |\nabla_{X_j} \psi| + \frac{c_2^2}{c^2} \left(\sum_{k \neq j} |\nabla_{X_k} \psi| \right) \\
&= \frac{c_1^2 - c_2^2}{c^2} |\nabla_{X_j} \psi| + \frac{c_2^2}{c^2} \sum_k |\nabla_{X_k} \psi|.
\end{aligned}$$

Thus, it follows

$$|\nabla_{X_j} \psi| \leqslant \frac{c_2^2}{c^2 - c_1^2 + c_2^2} \left(\sum_k |\nabla_{X_k} \psi| \right) = \frac{1}{n} \left(\sum_k |\nabla_{X_k} \psi| \right)$$

for every $j \in \{1, \dots, n\}$. This implies

$$|\nabla_{X_1} \psi| = |\nabla_{X_2} \psi| = \dots = |\nabla_{X_n} \psi|.$$

In dimension $n \geqslant 3$, we conclude that for any unit vector field X the function $|\nabla_X \psi|$ is independent of X. Therefore, if X and Y are orthogonal unit vector fields, then $\frac{1}{\sqrt{2}}(X+Y)$ is a unit vector field and we have the equations

$$\frac{1}{2}|\nabla_X \psi + \nabla_Y \psi|^2 = |\nabla_X \psi|^2 = |\nabla_Y \psi|^2,$$

which imply

$$\langle \nabla_X \psi, \nabla_Y \psi \rangle + \langle \nabla_Y \psi, \nabla_X \psi \rangle = 0. \tag{82}$$

This yields the following.

Remark 7.3. In the limiting case of (66) and also of (47), every associated eigenspinor ψ satisfies the equation

$$|Y||\nabla_X \psi| = |X||\nabla_Y \psi| \tag{83}$$

for all vector fields X, Y. Moreover, for all X, Y with $X \perp Y$, Eq. (82) is valid.

We consider now the limiting case of the improved version of the estimate (53), which is given by

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S_0 + \frac{4\nu_0^2}{(\sqrt{\eta + 2b\nu_0} + \sqrt{\eta})^2}\right). \tag{84}$$

By the proof of Theorem 5.1 we see that, in the limiting case with $\eta > 0$ ($\delta W \neq 0$), every associated eigenspinor ψ satisfies the equations $\mathcal{P}^s \psi = 0$ and $Q^t \psi = 0$ with $s = \frac{1-r}{r} t$, where r and t denote the optimal parameters here. The latter equation is equivalent to the equations

$$\mathcal{D}_X \psi = t B\big(X, X^k\big) \cdot \nabla_{X_k} \psi. \tag{85}$$

Using (85) we calculate

$$|\mathcal{D}\psi|^2 = t^2 \big|\big\langle B\big(X_j, X^k\big) \cdot \nabla_{X_k}\psi, B\big(X^j, X^l\big) \cdot \nabla_{X_l}\psi\big\rangle\big| = t^2 \big|\big\langle B^2\big(X^k, X^l\big) \nabla_{X_k}\psi, \nabla_{X_l}\psi\big\rangle\big| = t^2 b^2 |\mathcal{D}\psi|^2.$$

The latter equation is valid by the inequality (60), which must be an equality in the limiting case. This yields $\mathcal{D}\psi = 0$ or $t = \frac{1}{b}$. $\mathcal{D}\psi = 0$ implies that the manifold is Einstein and, hence, the contradiction $\eta = 0$. On the other hand, if $t = \frac{1}{b}$ is the optimal parameter, then comparing with (41) we see that this is the optimal parameter in the W-harmonic case. And this is also a contradiction. Hence, the following assertion is valid.

Remark 7.4. Every limiting manifold of the estimate (84) and also of the estimate (53) is W-harmonic and, hence, R-harmonic since S must be constant in the limiting case.

According to this it remains to study the limiting case of the estimate

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\left(S + \frac{2\nu_0}{b}\right) \tag{86}$$

for R-harmonic manifolds. Then we know that every associated eigenspinor ψ satisfies Eqs. (85), which by (6) and $t = \frac{1}{b}$ are equivalent to

$$\mathcal{D}_X \psi = \frac{1}{b} B\big(X, X^k\big) \cdot \mathcal{D}_{X_k} \psi. \tag{87}$$

We remark that in the limiting case of (86) every associated eigenspinor ψ also satisfies the eigenvalue equation

$$H\psi = \nu_0 \psi. \tag{88}$$

Proposition 7.2. *If M is a compact R-harmonic Riemannian spin manifold admitting a global spinor field $\psi \neq 0$ that satisfies Eqs. (87) and (88), then M is a limiting manifold of the estimate (86).*

Proof. Eqs. (87) written in the form

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = \frac{1}{b} B\big(X, X^k\big) \cdot \nabla_{X_k} \psi \tag{89}$$

imply

$$\nabla^2_{X,Y} \psi + \frac{1}{n} Y \cdot \nabla_X D\psi = \frac{1}{b}\big((\nabla_X B)\big(Y, X^k\big) \cdot \nabla_{X_k}\psi + B\big(Y, X^k\big) \cdot \nabla^2_{X, X_k}\psi\big). \tag{90}$$

Using $\delta B = 0$ we calculate

$$\begin{aligned}
\frac{n-1}{n} D^2\psi - \frac{S}{4}\psi &= D^2\psi - \frac{S}{4}\psi - \frac{1}{n} D^2\psi \\
&\overset{(22)}{=} -\nabla_{X^j,X_j}\psi - \frac{1}{n} X^j \cdot \nabla_{X_j} D\psi \overset{(90)}{=} -\frac{1}{2b} B\big(X^j, X^k\big) \cdot C(X_j, X_k)\psi \\
&\overset{(5),\,(6)}{=} \frac{1}{2b} B\big(X^k, X^j\big) \cdot B(X_j, X_k)\psi = \frac{1}{2b} H\psi \overset{(88)}{=} \frac{\nu_0}{2b}\psi.
\end{aligned}$$

This yields the eigenvalue equation

$$D^2\psi = \frac{n}{4(n-1)}\left(S + \frac{2\nu_0}{b}\right)\psi,$$

which shows that M is a limiting manifold. □

We remark that (87) implies

$$\mathcal{D}_X\psi = \frac{1}{b^2} B^2\big(X^k, X\big) \cdot \mathcal{D}_{X_k}\psi. \tag{91}$$

Now, a calculation with (91) similar to that with (81) yields the following assertion.

Remark 7.5. In the limiting case of the estimate (86) and also of the estimate (42), every associated eigenspinor ψ satisfies the equation

$$|Y||\mathcal{D}_X\psi| = |X||\mathcal{D}_Y\psi| \tag{92}$$

for all vector fields X, Y. Moreover, for all X, Y with $X \perp Y$ the equation

$$\langle \mathcal{D}_X\psi, \mathcal{D}_Y\psi\rangle + \langle \mathcal{D}_Y\psi, \mathcal{D}_X\psi\rangle = 0 \tag{93}$$

is valid.

The only known solutions of Eqs. (82), (83) or (92), (93) are twistor-spinors and, hence, Killing spinors in the limiting case. This leads to the conjecture that in the considered limiting cases the associated eigenspinors are Killing spinors. A classification of manifolds with real Killing spinors is given in [3]. Moreover, a summary of essential results concerning the spectrum of the Dirac operator one finds in [7].

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